Math 53: Multivariable Calculus **Sections 102, 103**

Worksheet for 2020-01-27

Conceptual questions

Question 1. What is the shape of the polar curve $r = 4\sin(\theta)$? Give bounds for θ so that this shape is traced exactly once. Hint: you can convert it to a Cartesian equation more easily if you multiply both sides by r.

Question 2. Is the polar curve $r = \cos(2\theta)$ symmetric about the line $y = x$?

Question 3. Suppose that two polar curves $r = f(\theta)$ and $r = g(\theta)$ are defined for $0 \le \theta \le 2\pi$. If the curves are identical (in the xy-plane), can you conclude that f and g are the same function?

What if the two curves were defined for $0 \le \theta < \pi$ instead?

Computations

Problem 1. Find the slope of the tangent line to the polar curve $r = 1/\theta$ at the point where $\theta = \pi$.

Problem 2 (Stewart §10.3.64)**.** Find all points on the polar curve $r = e^\theta$ where the tangent line is either horizontal or vertical.

Problem 3. Find a polar equation $r = f(\theta)$ for the circle centered at the point (a, b) (given in Cartesian coordinates) passing through the origin[1]. Give bounds for θ so that the circle is traced only once.

For the particular case $a = 1, b = 1$, do you get the same answer as what we got yesterday: $r = 2\sqrt{2}\cos(\theta - \pi/4)$?

Problem 4. This problem is about the polar curve $r = 2 + \cos(3\theta/2)$ graphed in Figure 1. This curve also appears in Stewart §10.3.54 (a matching problem).

(a) Find the Cartesian coordinates of the three points of self-intersection.
(b) At the self-intersection point in the first quadrant, there are *two* tangent lines. Find their equations.

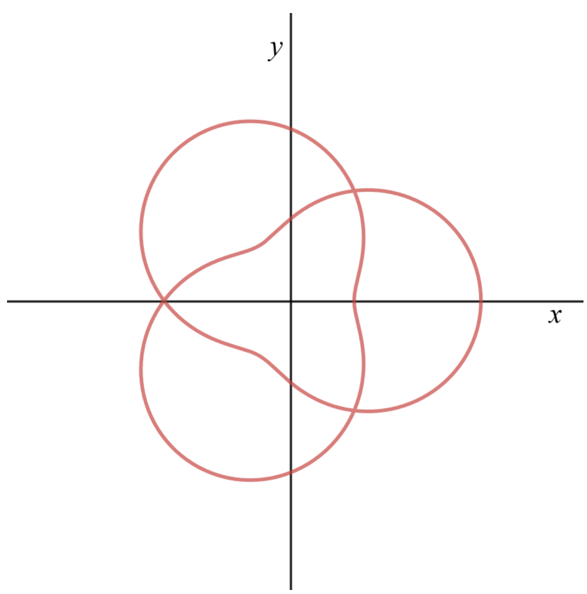

FIGURE 1. The polar curve $r = 2 + \cos(3\theta/2)$.

[1] The only circles which have a "nice" polar form are those centered at the origin or passing through the origin.

Below are the numerical answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to conceptual questions

Question 1. As θ goes from 0 to π, this traces out the circle centered at the point $(0, 2)$ with radius 2. (Complete the square).

Question 2. Yes. Try drawing the curve, either manually or e.g. with Desmos.

Question 3. $r = 1$ and $r = -1$ both draw out the circle of radius 1 centered at the origin as θ goes from 0 to 2π so the answer to the first question is "no."

The answer to the second question is yes, essentially because if you restrict θ to be between 0 and π, there is only one polar representative for each point in the Cartesian plane (other than the origin).

Answers to computations

Problem 1. $-\pi$. (Switch to parametric as $x = (1/\theta)\cos\theta$, $y = (1/\theta)\sin\theta$.)

Problem 2. The θ values for vertical tangents occur when $\theta = \pi/4 + k\pi$, where k is an integer. For horizontal tangents: $\theta = 3\pi/4 + k\pi$.

I didn't specify whether to give your answers in polar or Cartesian, but here are the points in Cartesian anyway:

$$\left((-1)^k\frac{\sqrt{2}}{2}e^{\pi/4+k\pi}, (-1)^k\frac{\sqrt{2}}{2}e^{\pi/4+k\pi}\right)$$

and

$$\left((-1)^{k+1}\frac{\sqrt{2}}{2}e^{3\pi/4+k\pi}, (-1)^k\frac{\sqrt{2}}{2}e^{3\pi/4+k\pi}\right)$$

Problem 3. $r = 2a\cos\theta + 2b\sin\theta$. (Write out the equation in Cartesian and then substitute.)

If we plug in $a = 1$, $b = 1$, we get $r = 2\cos\theta + 2\sin\theta$. This is the same as $2\sqrt{2}\cos(\theta - \pi/4)$ by the angle subtraction formula for cosine.

Problem 4.

(a) $(-2, 0)$, $(1, \sqrt{3})$, $(1, -\sqrt{3})$. (Deduce the θ values either from rotational symmetry or by noting that, if the curve reaches an intersection point at some θ, then it reaches it again at $\theta + 2\pi$. Then solve $2 + \cos(3\theta/2) = 2 + \cos(3(\theta + 2\pi)/2)$.)

(b) Here are the equations:

$$y - \sqrt{3} = \left(\frac{16}{13} - \frac{25}{39}\sqrt{3}\right)(x - 1)$$
$$y - \sqrt{3} = \left(-\frac{16}{13} - \frac{25}{39}\sqrt{3}\right)(x - 1)$$

(Use the slope formula; first plug in $\theta = \pi/3$ and then $\theta = 7\pi/3$ because those correspond to the different "branches" at the intersection point.)